Discipline: **Physics**
Subject: **Electromagnetic Theory**
Unit 16:
Lesson/ Module: **Theory of Relativity - III**

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Contents

Learning Objectives:

From this module students may get to know about the following:

1. *Covariant form of the equations of electrodynamics, which provides the link between electrodynamics and the theory of relativity.*
2. *The electric and magnetic field together as an antisymmetric four tensor.*
3. *Maxwell's equations written clearly in a covariant form.*
4. *Transformation properties of electric and magnetic fields under Lorentz transformations.*
5. *Derivation of the field of a moving charge as an example of these transformations.*

I6. Theory of relativity - III

16.1 Covariance of electrodynamics

We now discuss the covariance of the equations of electrodynamics. By this we mean that the various equations governing the electromagnetic phenomena have the same form in all inertial frames of reference under Lorentz transformations. Remember, this is from where the whole thing started. Lorentz and Poincare had already shown, even before the advent of Einstein's theory of relativity that equations of electrodynamics are not invariant under Galilean transformations; they are invariant under transformations which we now call the *Lorentz transformations.* In three dimensional ordinary space, the invariance of the laws of physics is best demonstrated by writing them as relations between scalars, vectors and tensors etc, though it is by no means necessary to do so. In a similar manner the invariance of the laws of electrodynamics under Lorentz transformations is best demonstrated by writing them as relations between quantities having definite transformation properties under these transformations, i.e., as relations between *four-scalars*, *four vectors* or *four-tensors* etc. This is what we mean by "covariance". We know that the equations of electrodynamics are invariant; however, it is fruitful to put them in an explicitly covariant form.

Let us first rewrite the relevant equations of electrodynamics. The Maxwell equations

$$\vec{\nabla}.\vec{E} = \rho / \varepsilon_0 , \tag{1}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \varepsilon_0 \partial \vec{E} / \partial t) \tag{2}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 , \tag{3}$$

$$\vec{\nabla}.\vec{B} = 0 \tag{4}$$

The Lorentz force equation for the force on a charged particle of mass m and charge q and moving with velocity $\vec{v}$ is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{5}$$

and the equation of continuity is

$$\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0 . \tag{6}$$

Apart from the electric and magnetic fields, two other quantities appear in these equations, the speed of light, c, and the charge q. We already know from the second postulate of the special theory that speed of light is an invariant; it is the same in all frames. Experiments show that charge is also an invariant. This has been established to a very high degree of accuracy.

Consider the volume element d^4x in our four-dimensional space-time defined as

$$d^4x = dx^0 dx^1 dx^2 dx^3 \qquad (7)$$

This transforms as follows under a Lorentz transformation

$$d^4x' = dx'^0\, dx'^1\, dx'^2\, dx'^3 = |J| dx^0 dx^1 dx^2 dx^3 = |J| d^4x\,. \qquad (8)$$

Here $|J|$ is the Jacobian of the Lorentz transformation matrix. For the special Lorentz transformation when the relative velocity of the two inertial frames is along the *x*-axis, the matrix is

$$\Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (9)$$

and $|J|$ equals unity. Hence $d^4x' = d^4x$ - the volume element is an invariant. If the charge density is ρ and since the element of charge

$$dq = \rho dx^1 dx^2 dx^3 \qquad (10)$$

is an invariant, the charge density must transform as the time component of a 4-vector. Keeping in mind the definition of a 4-divergence, $\partial_\alpha A^\alpha = \partial_0 A^0 + \vec{\nabla}.\vec{A}$, it is now natural to propose that ρ and $\vec{J}$ together form a four vector J^α:

$$J^\alpha = (c\rho, \vec{J})\,. \qquad (11)$$

The continuity equation (6) then takes the obviously covariant form

$$\partial_\alpha J^\alpha = 0. \qquad (12)$$

That J^α is a legitimate 4-vector thus follows from the invariance of electric charge.

For the Lorentz gauge, the equations for the vector potential $\vec{A}$ and the scalar potential Φ take the form of wave equations and are

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\rho/\varepsilon_0 \qquad (13)$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \qquad (14)$$

Keeping in mind the definition of the 4-dimensional Laplacian

$$\square = \partial^{\alpha}\partial_{\alpha} = \frac{\partial^2}{\partial x^{0^2}} - \vec{\nabla}^2 , \tag{15}$$

the above equations can be written as

$$\square \vec{A} = \mu_0 \vec{J} \tag{16}$$

$$\square \Phi = \frac{1}{c\varepsilon_0}(c\rho) \tag{17}$$

The Lorentz gauge condition that the potentials $(\vec{A}, \Phi)$ satisfy is

$$\frac{1}{c^2}\frac{\nabla\Phi}{\partial t} + \vec{\nabla}.\vec{A} = 0 . \tag{18}$$

Since $(c\rho, \vec{J})$ together form a 4-vector, requirement of Lorentz invariance demands that the potentials $(\Phi, \vec{A})$ too form a 4-vector,

$$A^{\alpha} = (\Phi / c, \vec{A}) . \tag{19}$$

Then the wave equations and the Lorentz gauge condition take on the manifestly covariant form

$$\square A^{\alpha} = \mu_0 J^{\alpha} , \tag{20}$$

$$\partial_{\alpha} A^{\alpha} = 0 . \tag{21}$$

16.1.1 The Field-strength tensor

Now the fields $\vec{E}$ and $\vec{B}$ are expressed in terms of the potentials as

$$\begin{aligned} \vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned} . \tag{22}$$

The x-components of $\vec{E}$ and $\vec{B}$ are explicitly

$$
\begin{aligned}
E_x &= -\frac{\partial A_x}{\partial t} - \frac{\partial \Phi}{\partial x} = -(\partial^0 cA^1 - \partial^1 cA^0) \\
B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -(\partial^2 A^3 - \partial^3 A^2)
\end{aligned}
\qquad (23)
$$

Here we have used equation (19) to express A^α in terms of $(\Phi, \vec{A})$. In a similar manner we can write the *y* and *z* components also:

$$
\begin{aligned}
E_y &= -\frac{\partial A_y}{\partial t} - \frac{\partial \Phi}{\partial y} = -(\partial^0 cA^2 - \partial^2 cA^0) \\
B_y &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = -(\partial^3 A^1 - \partial^1 A^3)
\end{aligned}
\qquad (24)
$$

$$
\begin{aligned}
E_z &= -\frac{\partial A_z}{\partial t} - \frac{\partial \Phi}{\partial z} = -(\partial^0 cA^1 - \partial^1 cA^0) \\
B_z &= \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -(\partial^1 A^2 - \partial^2 A^1)
\end{aligned}
\qquad (25)
$$

It is clear from these equations that $\vec{E}$ and $\vec{B}$ *do no transform as components of a 4-vector*. Each of these components of $\vec{E}$ and $\vec{B}$ is anti-symmetric in the indices appearing in the expressions on the right hand side. In fact the six components of $\vec{E}$ and $\vec{B}$ together form an anti-symmetric tensor of rank two:

$$
F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \, . \qquad (26)
$$

This tensor is called the *field-strength tensor*.

A tensor of rank two has sixteen independent components. For a symmetric tensor, $F^{\alpha\beta} = F^{\alpha\beta}$, and we are left with only ten independent components. For an anti-symmetric tensor on the other hand, $F^{\alpha\beta} = -F^{\alpha\beta}$, which means the diagonal elements must all be zero and we are left with only six independent elements.

Writing the various components of this tensor (26) and comparing with the expressions (23)-(25) above, we have

$$
F^{\alpha\beta} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \qquad (27)
$$

Once the contravariant field-strength tensor is defined, one can easily obtain the covariant or the mixed form of the field-strength tensor by using the metric tensor. For example,

$$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \qquad (28)$$

Notice that the contravariant and covariant forms differ only in the sign of electric field terms, the magnetic field terms remaining unchanged: In going from contravariant to covariant form, $(\vec{E}, \vec{B}) \to (-\vec{E}, \vec{B})$.

Another useful quantity is the *dual field-strength tensor* $f^{\alpha\beta}$. Before introducing this tensor, we introduce another quantity, a totally anti-symmetric tensor of rank four $\varepsilon^{\alpha\beta\gamma\delta}$, defined by

$$\varepsilon^{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{for } \alpha=0, \beta=1, \gamma=2, \delta=3 \text{ and any even permutation} \\ -1 & \text{for any odd permutation of the above symbols} \\ 0 & \text{if any two or more symbols are same.} \end{cases}$$

Since all four symbols have to different, only 4! = 24 elements are different from zero, of which 12 are positive and 12 are negative. It is easy to verify by direct evaluation that $\varepsilon^{\alpha\beta\gamma\delta} = -\varepsilon_{\alpha\beta\gamma\delta}$. This also implies that the tensor $\varepsilon^{\alpha\beta\gamma\delta}$ is a *pseudotensor* under spatial inversions.

The dual field-strength tensor is defined by

$$f^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3/c & -E_2/c \\ B_2 & -E_3/c & 0 & E_1/c \\ B_3 & E_2/c & -E_1/c & 0 \end{pmatrix} \qquad (29)$$

Comparing expressions (27) and (29) for $F^{\alpha\beta}$ and $f^{\alpha\beta}$, we see that the elements of the dual field-strength tensor $f^{\alpha\beta}$ are obtained from those of $F^{\alpha\beta}$ by the replacement $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$

16.1.2 Maxwell's equations

We now come to the Maxwell equations themselves which obviously must also be put in a manifestly covariant form. The two inhomogeneous equations are

$$\vec{\nabla}.\vec{E} = \rho / \varepsilon_0 , \tag{30a}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J} + \varepsilon_0 \partial \vec{E} / \partial t) \tag{30b}$$

These can be conveniently put in a covariant form by using the field-tensor $F^{\alpha\beta}$ and the current 4-vector $\vec{J}^{\alpha}$:

$$\partial_{\alpha} F^{\alpha\beta} = \mu_0 J^{\beta} \tag{31}$$

These can be verified directly by choosing β = (0, 1, 2, 3) and using equation (27) for $F^{\alpha\beta}$ Similarly the two homogenous Maxwell's equations

$$\begin{aligned} &\vec{\nabla}.\vec{B} = 0 \\ &\vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \end{aligned} \tag{32}$$

can be written in terms of the dual field tensor, $f^{\alpha\beta}$, as

$$\partial_{\alpha} f^{\alpha\beta} = 0 . \tag{33}$$

These equations can also be written in terms of the field-tensor $F^{\alpha\beta}$ by using the relation (29) between the two and take the form
.

$$\partial^{\alpha} F^{\beta\gamma} + \partial^{\beta} F^{\gamma\alpha} + \partial^{\gamma} F^{\alpha\beta} = 0 \tag{34}$$

All that is left now is to put the Lorentz force equation for a particle of charge q,

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{u} \times \vec{B}) \tag{35}$$

also in a covariant form. For this purpose we define two more 4-vectors. Just as we have the 4-velocity vector U^{α}, we define *acceleration 4-vector* a^{α} as

$$a^{\alpha} = \frac{dU^{\alpha}}{d\tau} . \tag{36}$$

The space part of this 4-vector represents the ordinary acceleration in the non-relativistic limit ($\gamma \to 1$). We also note in passing that the 4-acceleration and 4-velocity are orthogonal to each other:

$$a^{\alpha}U_{\alpha} = \frac{dU^{\alpha}}{d\tau}U_{\alpha} = \frac{1}{2}\frac{d}{d\tau}(U^{\alpha}U_{\alpha}) = \frac{1}{2}\frac{d}{d\tau}(c^2) = 0 .$$

From equation (36) it is now straightforward to define *4-force* F^{α} as*:*

$$F^{\alpha} = \frac{dP^{\alpha}}{d\tau} = m\frac{dU^{\alpha}}{d\tau} = ma^{\alpha} . \qquad (37)$$

Now the Lorentz 4-force vector should involve the electric and magnetic fields through $F^{\alpha\beta}$ and the velocity through U^{α}. In addition it should be linear in both $F^{\alpha\beta}$ and U^{α}. It thus follows that the Lorentz force is given by

$$F^{\alpha} = \frac{dP^{\alpha}}{d\tau} = m\frac{dU^{\alpha}}{d\tau} = ma^{\alpha} = qF^{\alpha\beta}U_{\beta} . \qquad (38)$$

Let us write the "1" component of this 4-vector equation:

$$F^1 = \gamma\frac{dp_x}{dt} = qF^{1\beta}U_{\beta} = q(F^{10}U_0 + F^{11}U_1 + F^{12}U_2 + F^{13}U_3)$$
$$= q(E_x\gamma - B_x\gamma u_x + B_y\gamma u_z)$$

or

$$\frac{dp_x}{dt} = q[E_x + (\vec{u}\times\vec{B})_x] \qquad (39)$$

In a similar fashion we can work out the other two components of this equation to obtain

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{u}\times\vec{B}) , \qquad (40)$$

the Lorentz force equation. Now let us look at the time component:

$$\frac{dP^0}{d\tau} = m\frac{dU^0}{d\tau} = m\gamma\frac{d(\gamma c)}{dt} = \frac{\gamma}{c}\frac{dW}{dt} = qF^{0\beta}U_{\beta} = \frac{q}{c}(E_x\gamma u_x + E_y\gamma u_y + E_z\gamma u_z) = \frac{\gamma q}{c}\vec{E}.\vec{u}$$

or

$$\frac{dW}{dt} = q\vec{E}.\vec{u} \qquad (41)$$

This is just the law of conservation of energy. Whereas the left hand side is the rate of increase of mechanical energy of the system, the right hand is the rate at which the electric field does work on the particle.

16.2 Transformation property of the electric and magnetic fields

As we have demonstrated, the electric and magnetic fields are not the space components of 4-vectors; rather the two of them together form components of a anti-symmetric 4-tensor of rank two. The transformation properties of the fields are therefore those of a tensor of rank two under Lorentz transformations, viz.,

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \qquad (42)$$

The matrix of Lorentz transformation is defined through

$$\Lambda^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \qquad (43)$$

Then the above equation can be put in the matrix form

$$F'^{\alpha\beta} = \Lambda^{\alpha}{}_{\gamma} F^{\gamma\delta} \Lambda^{\beta}{}_{\delta} = (\Lambda F \tilde{\Lambda})^{\alpha\beta}$$

or

$$F' = \Lambda F \Lambda^{T} \qquad (44)$$

Here Λ^{T} refers to the transpose of Λ. For the case when the two coordinate systems are aligned and the relative velocity is in the *x*-direction, the Lorentz transformation matrix is given by

$$\Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (45)$$

Using expression (27) for $F^{\alpha\beta}$ and the corresponding expression for $F'^{\alpha\beta}$, we have

$$\begin{pmatrix} 0 & -E'_1/c & -E'_2/c & -E'_3/c \\ E'_1/c & 0 & -B'_3 & B'_2 \\ E'_2/c & B'_3 & 0 & -B'_1 \\ E'_3/c & -B'_2 & B'_1 & 0 \end{pmatrix} =$$

$$\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying the three matrices on the right hand side and comparing term by term, we obtain the desired relation between the components of the electric and magnetic fields in two inertial frames of reference:

$$\begin{aligned} E'_1 &= E_1 & B'_1 &= B_1 \\ E'_2 &= \gamma(E_2 - \beta c B_3) & B'_2 &= \gamma(B_2 + \beta E_3/c) \\ E'_3 &= \gamma(E_3 + \beta c B_2) & B'_3 &= \gamma(B_3 - \beta E_2/c) \end{aligned} \tag{46}$$

In these relations the subscripts *1, 2, 3* are ordinary Cartesian indices and not covariant indices, i.e., *1, 2, 3* refer to the *x, y, z* components of the ordinary vectors of electric and magnetic fields. As usual, the inverse transformation from the primed to the unprimed frame is obtained by interchanging primed and unprimed quantities and the replacement $\beta \to -\beta$:

$$\begin{aligned} E_1 &= E'_1 & B_1 &= B'_1 \\ E_2 &= \gamma(E'_2 + \beta c B'_3) & B_2 &= \gamma(B'_2 - \beta E'_3/c) \\ E_3 &= \gamma(E'_3 - \beta c B'_2) & B_3 &= \gamma(B'_3 + \beta E'_2/c) \end{aligned} \tag{47}$$

For the case when the axes of the two frames are aligned, but the relative velocity $\vec{v}$ is in a general direction

$$\begin{aligned} t' &= \gamma\left(t - \frac{\vec{v}.\vec{x}}{c^2}\right) = \gamma\left(t - \frac{\vec{\beta}.\vec{x}}{c}\right) \\ \vec{x}' &= \vec{x} + \frac{\gamma-1}{v^2}(\vec{v}.\vec{x})\vec{v} - \gamma\vec{v}t = \vec{x} + \frac{\gamma-1}{\beta^2}(\vec{\beta}.\vec{x})\vec{\beta} - \gamma\vec{\beta}t/c \end{aligned} \tag{48}$$

In terms of space-time coordinates $x^\mu = (ct, \vec{x}) = (ct, x_1, x_2, x_3)$, the above equations can be written as

$$\begin{bmatrix} x'_0 \\ x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\frac{(\gamma-1)\beta_1^{\,2}}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_2}{\beta^2} & \frac{(\gamma-1)\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & \frac{(\gamma-1)\beta_1\beta_2}{\beta^2} & 1+\frac{(\gamma-1)\beta_2^{\,2}}{\beta^2} & \frac{(\gamma-1)\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & \frac{(\gamma-1)\beta_1\beta_3}{\beta^2} & \frac{(\gamma-1)\beta_2\beta_3}{\beta^2} & 1+\frac{(\gamma-1)\beta_3^{\,2}}{\beta^2} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In other words, the matrix of the corresponding Lorentz transformation is

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\dfrac{(\gamma-1)\beta_1{}^2}{\beta^2} & \dfrac{(\gamma-1)\beta_1\beta_2}{\beta^2} & \dfrac{(\gamma-1)\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & \dfrac{(\gamma-1)\beta_1\beta_2}{\beta^2} & 1+\dfrac{(\gamma-1)\beta_2{}^2}{\beta^2} & \dfrac{(\gamma-1)\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & \dfrac{(\gamma-1)\beta_1\beta_3}{\beta^2} & \dfrac{(\gamma-1)\beta_2\beta_3}{\beta^2} & 1+\dfrac{(\gamma-1)\beta_3{}^2}{\beta^2} \end{bmatrix} \qquad (49)$$

Using this expression for the transformation matrix we obtain the relation between the components in the general case:

$$\begin{pmatrix} 0 & -E'_1/c & -E'_2/c & -E'_3/c \\ E'_1/c & 0 & -B'_3 & B'_2 \\ E'_2/c & B'_3 & 0 & -B'_1 \\ E'_3/c & -B'_2 & B'_1 & 0 \end{pmatrix} =$$

$$\begin{bmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\dfrac{(\gamma-1)\beta_1{}^2}{\beta^2} & \dfrac{(\gamma-1)\beta_1\beta_2}{\beta^2} & \dfrac{(\gamma-1)\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & \dfrac{(\gamma-1)\beta_1\beta_2}{\beta^2} & 1+\dfrac{(\gamma-1)\beta_2{}^2}{\beta^2} & \dfrac{(\gamma-1)\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & \dfrac{(\gamma-1)\beta_1\beta_3}{\beta^2} & \dfrac{(\gamma-1)\beta_2\beta_3}{\beta^2} & 1+\dfrac{(\gamma-1)\beta_3{}^2}{\beta^2} \end{bmatrix}$$

$$\begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & -B_3 & B_2 \\ E_2/c & B_3 & 0 & -B_1 \\ E_3/c & -B_2 & B_1 & 0 \end{pmatrix}$$

$$\begin{bmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1+\dfrac{(\gamma-1)\beta_1{}^2}{\beta^2} & \dfrac{(\gamma-1)\beta_1\beta_2}{\beta^2} & \dfrac{(\gamma-1)\beta_1\beta_3}{\beta^2} \\ -\gamma\beta_2 & \dfrac{(\gamma-1)\beta_1\beta_2}{\beta^2} & 1+\dfrac{(\gamma-1)\beta_2{}^2}{\beta^2} & \dfrac{(\gamma-1)\beta_2\beta_3}{\beta^2} \\ -\gamma\beta_3 & \dfrac{(\gamma-1)\beta_1\beta_3}{\beta^2} & \dfrac{(\gamma-1)\beta_2\beta_3}{\beta^2} & 1+\dfrac{(\gamma-1)\beta_3{}^2}{\beta^2} \end{bmatrix}$$

On comparing the two sides of the equation, term by term, we obtain the desired result:

$$
\begin{aligned}
E'_1 &= \gamma[E_1 + (\beta_2 cB_3 - \beta_3 cB_2) - \frac{\gamma^2}{\gamma+1}(\beta_1 E_1 + \beta_2 E_2 + \beta_3 E_3)\beta_1] \\
E'_2 &= \gamma[E_2 + (\beta_3 cB_1 - \beta_1 cB_3) - \frac{\gamma^2}{\gamma+1}(\beta_1 E_1 + \beta_2 E_2 + \beta_3 E_3)\beta_2] \\
E'_3 &= \gamma[E_3 + (\beta_1 cB_2 - \beta_2 cB_1) - \frac{\gamma^2}{\gamma+1}(\beta_1 E_1 + \beta_2 E_2 + \beta_3 E_3)\beta_3] \\
B'_1 &= \gamma[B_1 - (\beta_2 E_3 - \beta_3 E_2)/c - \frac{\gamma^2}{\gamma+1}(\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3)\beta_1] \\
B'_2 &= \gamma[B_2 - (\beta_3 E_1 - \beta_1 E_3)/c - \frac{\gamma^2}{\gamma+1}(\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3)\beta_2] \\
B'_3 &= \gamma[B_3 - (\beta_1 E_2 - \beta_2 E_1)/c - \frac{\gamma^2}{\gamma+1}(\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3)\beta_3]
\end{aligned}
\tag{50}
$$

In the vector form the equations look much simpler

$$
\begin{aligned}
\vec{E}' &= \gamma(\vec{E} + c\vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}(\vec{\beta}.\vec{E})\vec{\beta} \\
\vec{B}' &= \gamma(\vec{B} - \vec{\beta} \times \vec{E}/c) - \frac{\gamma^2}{\gamma+1}(\vec{\beta}.\vec{B})\vec{\beta}
\end{aligned}
\tag{51}
$$

These equations clearly show that the electric and magnetic fields have no independent existence – they are intertwined with each other. Even if the field is pure electric or pure magnetic in one frame of reference, to an observer in a different frame of reference the field will have both components – electric as well as magnetic. Thus two are really integrated into a single object – the electromagnetic field.

The transformation of the fields is, however, not without any restrictions. As we have seen, given a tensor of certain rank, one can obtain tensors of lower ranks by contraction of indices. We are given two tensors of rank two involving the fields, viz., the field tensor $F^{\alpha\beta}$ and the dual field tensor $f^{\alpha\beta}$. From these we can form many invariants of which only two are independent:

$$F^{\alpha\beta}\ F_{\alpha\beta} \propto (B^2 - E^2) \tag{52}$$

$$F^{\alpha\beta}\ f_{\alpha\beta} \propto \vec{E}.\vec{B} \tag{53}$$

These being invariant, their values must be the same in all frames of reference. In particular, as a result, a purely electrostatic field cannot transform into a purely magnetostatic field in any other frame of reference. Further if the two fields are transverse to each other in one frame they will stay transverse in all frames. Also a pure electrostatic (magnetostatic) field in one frame will develop magnetostatic (electrostatic) component in other frames; the two components however will always be transverse.

16.4 Field of a uniformly moving charge

As an example of the transformation formulas developed above, we consider the field due to a uniformly moving charge. The result is well known from Biot-Savart law. In the frame of reference K a point charge q is moving with a uniform velocity $\vec{u}$ along the x-direction. The frame K' is moving along the common x-axis. The origin of space and time coordinates are chosen suitably so that at $t = t' = 0$, the origin of the two systems coincide. The location of the observer is given by the point P with coordinates (0,b,0) in K at $t = t' = 0$. The coordinates of the charge are given by (*ut, 0, 0).* In the frame K', the particle is at rest and is permanently located at the origin. The observer at P however is moving with speed u in the negative x- direction and its coordinates are given by **[See figure 11.8 Jackson Edition 2.]**

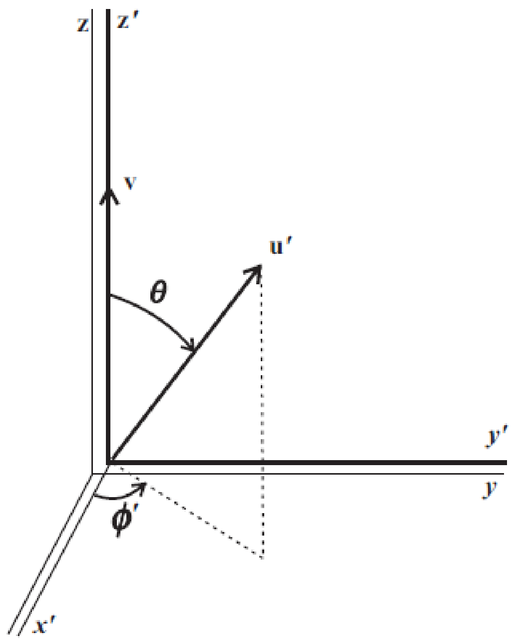


Fig: Addition of Velocities

$$x'_1 = -ut', x'_2 = b, x_3 = 0 \tag{54}$$

The distance between the point charge at (0,0,0) and the observer at ($-ut'$, b,0) is

$$r' = \sqrt{b^2 + (ut')^2} \tag{55}$$

The field in K' is purely electrostatic and is given by the coulomb law

$$\vec{E}' = \frac{q\vec{r}'}{4\pi\varepsilon_0 r'^3}, \quad \vec{B}' = 0\,. \tag{56}$$

In terms of components we have

$$E'_1 = -\frac{qut'}{4\pi\varepsilon_0 r'^3}, \qquad E'_2 = \frac{qut'}{4\pi\varepsilon_0 r'^3}, \qquad E'_3 = 0$$
$$B'_1 = 0, \qquad B'_2 = 0, \qquad B'_3 = 0, \tag{57}$$

To obtain the field in the unprimed frame we have to transform the fields as well as the coordinates from the primed to the unprimed frame. We need the transformation of time coordinate which is given by

$$t = \gamma(t' + \frac{u}{c^2}x'_1) = \gamma(t' - \frac{u^2}{c^2}t') = t'/\gamma\ . \tag{58}$$

Expressed in the coordinates of K, the field in K' is

$$E'_1 = -\frac{1}{4\pi\varepsilon_0}\frac{q\gamma ut}{[b^2 + (\gamma ut)^2]^{3/2}}, \quad E'_2 = \frac{1}{4\pi\varepsilon_0}\frac{qb}{[b^2 + (\gamma ut)^2]^{3/2}} \tag{59}$$

The other four components remain zero. The fields in the K frame are obtained by the inverse transformation (47)

$$E_1 = E'_1 \qquad B_1 = B'_1$$
$$E_2 = \gamma(E'_2 + \beta c B'_3) \qquad B_2 = \gamma(B'_2 - \beta E'_3/c)$$
$$E_3 = \gamma(E'_3 - \beta c B'_2) \qquad B_3 = \gamma(B'_3 + \beta E'_2/c)$$

Or

$$E_1 = E'_1 = -\frac{1}{4\pi\varepsilon_0}\frac{q\gamma ut}{[b^2 + (\gamma ut)^2]^{3/2}},$$
$$E_2 = \gamma E'_2 = \frac{1}{4\pi\varepsilon_0}\frac{\gamma qb}{[b^2 + (\gamma ut)^2]^{3/2}}$$
$$B_3 = \gamma\beta E'_2/c = \frac{1}{4\pi\varepsilon_0 c}\frac{\gamma qb\beta}{[b^2 + (\gamma ut)^2]^{3/2}} = \beta E_2/c$$

The other components are zero. The electric field is in the *x*-*y* plane and the magnetic field is normal to it in the *z*-direction. This is as expected. Since in the frame K' the field is purely electrostatic, $\vec{E}.\vec{B} = 0$ in K' and hence $\vec{E}.\vec{B} = 0$ in all frames. Further, since $\vec{B} = 0$ in K', $(E^2 - B^2) > 0$ in this frame and hence $(E^2 - B^2) > 0$ in all frames. This is also clearly true from the above equations.

At nonrelativistic speeds $\gamma \approx 1$, and

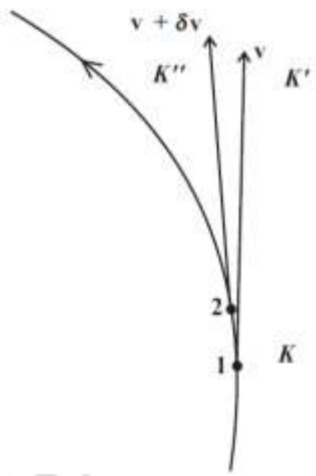
$$B_3 \approx \frac{u}{4\pi\varepsilon_0 c^2}\frac{qb}{r^3}$$

or

$$\vec{B} = \frac{\mu_0}{4\pi} q \frac{\vec{u} \times \vec{r}}{r^3}$$

This is the well-known Biot-Savart expression for the field of a moving charge.

On the other hand, at relativistic speeds, $\beta \to 1$, and the magnetic induction cB_3 becomes nearly equal to the transverse electric field E_2. The transverse electric field takes its maximum value at time $t = 0$, i.e., when the charge is closest to the point of observation. In the same limit, however, the duration of appreciable field strength at the point P is decreased. A measure of the time interval Δt over which the field remains appreciable is **[See Figure 11.9a Jackson Edition 2]**



$$\gamma^2 u^2 (\Delta t)^2 \approx b^2 \Rightarrow \Delta t \approx \frac{b}{\gamma u}$$

As γ increases, the peak fields E_2 and B_3 increase in proportion, but the duration for which the fields are appreciable goes in inverse proportion. The figure shows the behaviour of the transverse (E_2 and B_3) and longitudinal (E_1) fields as functions of time. For $\beta \to 1$ the observer at P sees transverse and mutually perpendicular electric and magnetic fields with electric field being nearly c times the magnetic field.. The extra longitudinal electric field varies rapidly from positive to negative values and has zero time integral, being an odd function of time. If our detecting apparatus has appreciable response time ($> \Delta t = \frac{b}{\gamma u}$), it will not respond to this longitudinal electric field. For all practical purposes the fields are transverse and mutually perpendicular.

Summary

1. *In this module the study of the special theory of relativity has been continued further. Covariant form of the equations of electrodynamics was derived. The covariance of the equations provides the necessary link between electrodynamics and the theory of relativity.*
2. *It is shown how electric and magnetic field together form an anti-symmetric tensor of rank two.*
3. *Maxwell's equations were written down in an explicitly covariant form.*
4. *From the electromagnetic field tensor the transformation properties of electric and magnetic fields under Lorentz transformations were derived. This demonstrated clearly that the electric and magnetic fields are to be regarded as one single entity.*
5. *The field of a moving charge was derived as an example of these transformations,*